

Milnor numbers in deformations of homogeneous singularities

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Abstract

Let f_0 be a plane curve singularity. We study the Milnor numbers of singularities in deformations of f_0 . We completely describe the set of these Milnor numbers for homogeneous singularities f_0 in the case of non-degenerate deformations and obtain some partial results on this set in the general case.

Keywords: deformation of singularity, Milnor number, Newton polygon, non-degenerate singularity

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Introduction

Let $f_0 : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ be an isolated singularity (in the sequel a singularity means an isolated singularity) and $\mu(f_0)$ its Milnor number at 0. Consider an arbitrary holomorphic deformation $(f_s)_{s \in S}$ of f_0 , where s is a single parameter defined in a neighborhood S of $0 \in \mathbb{C}$. By the semi-continuity (in the Zariski topology) of Milnor numbers in families of singularities $\mu(f_s)$ is constant for sufficiently small $s \neq 0$ and $\mu(f_s) \leq \mu(f_0)$. Denote this constant value by $\mu^{\text{gen}}(f_s)$ and call it *generic Milnor number of the deformation* (f_s) . Let

$$\mathcal{M}(f_0) = (\mu_0(f_0), \mu_1(f_0), \dots, \mu_k(f_0))$$

be the strictly decreasing sequence of generic Milnor numbers of all possible deformations of f_0 . In particular $\mu_0(f_0) = \mu(f_0) > \mu_1(f_0) > \dots > \mu_k(f_0) = 0$. If f_0 is a fixed singularity then the sequence $\mathcal{M}(f_0)$ will be denoted shortly $(\mu_0, \mu_1, \dots, \mu_k)$. Analogously we define

$$\mathcal{M}^{\text{nd}}(f_0) = (\mu_0^{\text{nd}}(f_0), \mu_1^{\text{nd}}(f_0), \dots, \mu_l^{\text{nd}}(f_0)),$$

the strictly decreasing sequence of generic Milnor numbers of all possible non-degenerate deformations of f_0 (it means that any element of the family (f_s) is a Kouchnirenko non-degenerate singularity). Notice that the sequence $\mathcal{M}^{\text{nd}}(f_0)$ is a subsequence of $\mathcal{M}(f_0)$. The problem of description of $\mathcal{M}(f_0)$ and $\mathcal{M}^{\text{nd}}(f_0)$ was posed by A. Bodin [Bod07] who, in turn, generalized related problems posed by A'Campo (unpublished) and V.I. Arnold [AGZV85] (Problems 1975-15, 1982-12). It is a non-trivial problem because by Gusein-Zade [GZ93], see also Brzostowski-Krasinski [BK14], Bodin [Bod07], Walewska [Wal10], [Wal13], the sequence $\mathcal{M}(f_0)$ and consequently $\mathcal{M}^{\text{nd}}(f_0)$ is not always equal to the sequence of all non-negative integers less than $\mu(f_0)$. For instance, $\mathcal{M}(x^4 + y^4) = (9, 7, 6, \dots, 1, 0)$, $\mathcal{M}^{\text{nd}}(x^4 + y^4) = (9, 6, 5, \dots, 1, 0)$. In the paper we will consider the class of homogeneous singularities in the plane. We describe completely the sequence $\mathcal{M}^{\text{nd}}(f_0)$ for homogeneous plane curve singularities f_0 and we give some partial results on $\mathcal{M}(f_0)$. The main results are:

Theorem 1 *If $f_0 : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0)$ is a homogeneous singularity of degree $d \geq 2$ (it means f_0 is a homogeneous polynomial of degree d without multiple factors) then*

$$\mathcal{M}^{\text{nd}}(f_0) = ((d-1)^2, (d-1)(d-2), \dots, 1, 0), \quad (1)$$

if d is odd or $d \leq 4$ or f_0 is non-convenient, and

$$\mathcal{M}^{\text{nd}}(f_0) = ((d-1)^2, (d-1)(d-2), \dots, \widehat{d^2 - 4d + 2}, \dots, 1, 0), \quad (2)$$

if d is even ≥ 6 and f_0 is convenient (\widehat{a} means the symbol a is omitted).

Remark 1 The value of the first jump in the above sequences $(d-1)^2 - (d-1)(d-2) = d-1$ has been given by A. Bodin [Bod07]. The value of the second one equal to 1 was established by J. Walewska [Wal10], [Wal13].

Remark 2 Since the Milnor number of a non-degenerate singularities depends only on its Newton diagram (see the Kouchnirenko Theorem in Preliminaries) we obtain that Theorem 1 holds also for semi-homogeneous singularities i.e. for singularities of the form $\tilde{f}_0 = f_0 + g$, where f_0 is a homogeneous singularity of degree d and g is a holomorphic function of order $> d$.

In case $\mathcal{M}(f_0)$ we can only complement the sequence $\mathcal{M}^{\text{nd}}(f_0)$ by some numbers.

Theorem 2 If $f_0 : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0)$ is a homogeneous singularity of degree $d \geq 3$ then

$$\mathcal{M}(f_0) = ((d-1)^2, \mu_1, \dots, \mu_r, (d-1)(d-2) + 1, (d-1)(d-2), \dots, 1, 0),$$

where μ_1, \dots, μ_r is an unknown subsequence (may be empty).

For a particular homogeneous singularity we have a more precise result.

Theorem 3 For the singularity $f_0(x, y) = x^d + y^d$, $d \geq 2$,

$$\mu_1(f_0) = (d-1)^2 - \left\lfloor \frac{d}{2} \right\rfloor, \quad (3)$$

where $\lfloor a \rfloor$ means the integer part of a real number a .

The value of $\mu_1(f_0)$ in Theorem 3, found for very specific singularities, could not be extend to the whole class of homogeneous singularities of degree d , because $\mathcal{M}(f_0)$ depends on coefficients of f_0 . Precisely we have

Theorem 4 For homogeneous singularities f_0 of degree $d \geq 2$

$$\mu_1(f_0) \leq (d-1)^2 - \left\lfloor \frac{d}{2} \right\rfloor$$

and for f_0 of degree $d \geq 5$ with generic coefficients we have

$$\mu_1(f_0) < (d-1)^2 - \left\lfloor \frac{d}{2} \right\rfloor. \quad (4)$$

Remark 3 Having generic coefficients means: there is a proper algebraic subset V in the space \mathbb{C}^{d+1} of coefficients of homogeneous polynomials of degree d such that for homogeneous singularities f_0 of degree d with coefficients outside V the inequality (4) holds.

1 Preliminaries

Let $f_0 : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ be an isolated singularity, i.e. f_0 is a germ of a holomorphic function having an isolated critical point at $0 \in \mathbb{C}^n$ and $0 \in \mathbb{C}$ as the corresponding critical value. A deformation of f_0 is the germ of a holomorphic function $f(z, s) : (\mathbb{C}^n \times \mathbb{C}, 0) \rightarrow (\mathbb{C}, 0)$ such that:

1. $f(z, 0) = f_0(z)$,
2. $f(0, s) = 0$.

The deformation $f(z, s)$ of the singularity f_0 will be treated as a family (f_s) of germs, taking $f_s(z) = f(z, s)$. Since f_0 is an isolated singularity, f_s has also isolated singularities near the origin, for sufficiently small s ([GLS07], Ch.I, Thm 2.6). Then the Milnor numbers $\mu(f_s)$ of f_s at 0 are defined. Since the Milnor number is upper semi-continuous in the Zariski topology in families of singularities ([GLS07], Prop. 2.57) there exists an open neighborhood S of $0 \in \mathbb{C}$ such that

1. $\mu(f_s) = \text{const}$ for $s \in S \setminus \{0\}$,
2. $\mu(f_0) \geq \mu(f_s)$ for $s \in S$.

Consequently, the notions of $\mu^{\text{gen}}(f_s)$, $\mathcal{M}(f_0)$, and $\mathcal{M}^{\text{nd}}(f_0)$ in the Introduction are well-defined.

Let \mathbb{N} be the set of nonnegative integers and \mathbb{R}_+ be the set of nonnegative real numbers. Let $f_0(x, y) = \sum_{(i,j) \in \mathbb{N}^2} a_{ij} x^i y^j$ be a singularity. Put $\text{Supp}(f_0) := \{(i, j) \in \mathbb{N}^2 : a_{ij} \neq 0\}$. The *Newton diagram* of f_0 is defined as the convex hull of the set $\bigcup_{(i,j) \in \text{Supp}(f_0)} ((i, j) + \mathbb{R}_+^2)$ and is denoted by $\Gamma_+(f_0)$. The boundary (in \mathbb{R}^2) of the diagram $\Gamma_+(f_0)$ is the sum of two half-lines and a finite number of compact line segments. The set of those line segments will be called *the Newton polygon* of f_0 and denoted by $\Gamma(f_0)$. For each segments $\gamma \in \Gamma(f_0)$ we define a weighted homogeneous polynomial $(f_0)_\gamma := \sum_{(i,j) \in \gamma} a_{ij} x^i y^j$. A singularity f_0 is called *non-degenerate* (in the Kouchnirenko sense) *on a segment* $\gamma \in \Gamma(f_0)$ if and only if the system of equations

$$\frac{\partial(f_0)_\gamma}{\partial x}(x, y) = 0, \quad \frac{\partial(f_0)_\gamma}{\partial y}(x, y) = 0,$$

has no solutions in $\mathbb{C}^* \times \mathbb{C}^*$. f_0 is called *non-degenerate* if and only if it is non-degenerate on every segment $\gamma \in \Gamma(f_0)$. A singularity is called *convenient* if $\Gamma_+(f_0)$ intersects both coordinate axes in \mathbb{R}^2 . For such singularities we denote by S the area of the domain bounded by the coordinate axes and the Newton polygon $\Gamma(f_0)$. Let a (resp. b) be the distance of the point $(0, 0)$ to the intersection of $\Gamma_+(f_0)$ with the horizontal (resp. vertical) axis. The number

$$v(f_0) := 2S - a - b + 1, \tag{K}$$

is called the *Newton number of the singularity* f_0 . Let us recall the Planar Kouchnirenko Theorem.

Theorem 5 ([Kou76]) *For a convenient singularity f_0 we have:*

1. $\mu(f_0) \geq v(f_0)$,
2. if f_0 is non-degenerate then $\mu(f_0) = v(f_0)$.

The Newton number of singularities is monotonic with respect to the Newton diagrams of these singularities (with the relation of inclusion).

Proposition 6 ([Len08], [GL07]) *If f_0 and \tilde{f}_0 are convenient singularities and $\Gamma_+(f_0) \subset \Gamma_+(\tilde{f}_0)$ then $v(f_0) \geq v(\tilde{f}_0)$.*

Corollary 7 *If f_0 and \tilde{f}_0 are convenient, non-degenerate singularities and $\Gamma_+(f_0) \subset \Gamma_+(\tilde{f}_0)$ then $\mu(f_0) \geq \mu(\tilde{f}_0)$.*

In the paper we will use „global” results concerning projective algebraic curves proved by A. Płoski.

Theorem 8 ([Plo14, Thm 1.1]) *Let $f = 0$, $f \in \mathbb{C}[X, Y]$, be a plane algebraic curve of degree $d > 1$ with an isolated singular point at $0 \in \mathbb{C}^2$. Suppose that $\text{ord}_0(f) < d$. Then*

$$\mu(f) \leq (d-1)^2 - \left\lfloor \frac{d}{2} \right\rfloor.$$

Remark 4 *The assumption $\text{ord}_0(f) < d$ in the above theorem means that f is not a homogeneous polynomial. If f is a homogeneous polynomial of degree d with an isolated singular point at $0 \in \mathbb{C}^2$ then obviously $\mu(f) = (d-1)^2$.*

Theorem 9 ([Plo14, Thm 1.4]) *Let f be a polynomial of degree $d > 2$, $d \neq 4$. Then the following two conditions are equivalent*

1. *The curve $f = 0$ passes through the origin and $\mu_0(f) = (d-1)^2 - [d/2]$,*
2. *The curve $f = 0$ has $d - [d/2]$ irreducible components. Each irreducible component of the curve passes through the origin. If $d \equiv 0 \pmod{2}$ then all components are of degree 2 and intersect pairwise at 0 with multiplicity 4. If $d \not\equiv 0 \pmod{2}$ then all but one component are of degree 2 and intersect pairwise at 0 with multiplicity 4, the remaining component is linear and tangent to all components of degree 2.*

2 Proof of Theorem 1

Let f_0 be a homogeneous isolated singularity of degree d i.e.

$$f_0(x, y) = a_0x^d + \dots + a_dy^d, a_i \in \mathbb{C}, d \geq 2, f_0 \neq 0$$

and f_0 has no multiple factors in $\mathbb{C}[x, y]$. Geometrically it is an ordinary singularity of d lines intersecting at the origin. Notice that f_0 is non-degenerate.

By A. Bodin [Bod07] and J. Walewska [Wal10] for any non-degenerate deformation (f_s) of f_0 , for which $\mu^{\text{gen}}(f_s) \neq \mu(f_0)$ we have $\mu^{\text{gen}}(f_s) \leq \mu(f_0) - (d-1) = (d-1)^2 - (d-1) = (d-1)(d-2) = d^2 - 3d + 2$.

A. Assume first that f_0 is convenient i.e. $a_0a_d \neq 0$. Since we consider only non-degenerate deformations of f_0 , by the Kouchnirenko Theorem we may assume that

$$f_0(x, y) = x^d + y^d, \quad d \geq 2. \quad (5)$$

We will apply induction with respect to the degree d . It is easy to find non-degenerate deformations (f_s) of f_0 for the degrees $d = 2, 3, 4$, whose generic Milnor numbers realize all the numbers $\leq d^2 - 3d + 2$. This gives

$$\mathcal{M}^{\text{nd}}(x^2 + y^2) = (1, 0),$$

$$\mathcal{M}^{\text{nd}}(x^3 + y^3) = (4, 2, 1, 0),$$

$$\mathcal{M}^{\text{nd}}(x^4 + y^4) = (9, 6, 5, \dots, 1, 0),$$

(in the last case one can use some of deformations given below). Let us consider singularity (5) where $d \geq 5$. It is easy to check (by the Kouchnirenko Theorem) that the deformations

1. $f_s = f_0 + sy^{d-1} + sx^{d-1}y^{2d-l-2}$ for $d-1 \leq l \leq 2d-3$ have generic Milnor numbers $d^2 - 4d + 4, \dots, d^2 - 3d + 2$, respectively,
2. $f_s = f_0 + sy^{d-2}$ has generic Milnor number $d^2 - 4d + 3$,
3. $f_s = f_0 + sy^{d-2}sx^{d-[\frac{d}{2}]}y^{[\frac{d}{2}]-1}$ has generic Milnor number $d^2 - 4d + 2$ for d odd and $d^2 - 4d + 3$ for d even.
4. $f_s = f_0 + sy^{d-1} + sx^ly^{d-l-2}$ for $1 \leq l \leq d-3$ have generic Milnor numbers $d^2 - 5d + 5, \dots, d^2 - 4d + 1$, respectively.

The above deformations „realize” all integers from $d^2 - 5d + 5$ to $d^2 - 3d + 2$ with exception of the number $d^2 - 4d + 2$ in the case d is even. Now we use induction hypothesis. Notice that for $(d-1)$ we have $((d-1) - 1)((d-1) - 2) = d^2 - 5d + 6 > d^2 - 5d + 5$. Hence, if d is odd then $(d-1)$ is even and by induction hypotheses we may „realize” all integers from 0 to $d^2 - 5d + 6$ with the exception of the number $(d-1)^2 - 4(d-1) + 2 = d^2 - 6d + 7$. But the deformation $f_s = f_0 + sy^{d-1} + sx^{d-5}y^2$ of f_0 has generic Milnor number equal to $d^2 - 6d + 7$. This gives formula (1).

If d is even, then $(d-1)$ is odd and by induction hypothesis we may find deformations of f_0 realizing all integers from 0 to $d^2 - 5d + 6$. Consequently in these cases we have found deformations of f_0 realizing all integers from 0 to $(d-1)(d-2)$ with the exception of the number $d^2 - 4d + 2$. Now we prove that this number is not generic Milnor number of any non-degenerate deformation of f_0 . Assume to the contrary that there exists a non-degenerate deformation (f_s) of $f_0(x, y) = x^d + y^d$, $d \geq 6$, d even, for which

$$\mu^{\text{gen}}(f_s) = d^2 - 4d + 2.$$

Since for sufficiently small $s \neq 0$ the Newton polygons of f_s are the same we consider the following cases:

I. $\text{ord}_{(x,y)} f_s \leq d-2$. Then there are points (i, j) in $\text{Supp} f_s$, $s \neq 0$, such that $i + j \leq d-2$. Take any such point (i, j) . Consider subcases:

Ia. $(i, j) \neq (0, d-2)$ and $(i, j) \neq (d-2, 0)$. Consider the non-degenerate auxiliary deformation of f_0

$$\tilde{f}_s(x, y) := \begin{cases} f_0(x, y) + sx^i y^{d-2-i} & \text{if } i > 0, \\ f_0(x, y) + sxy^{d-3} & \text{if } i = 0. \end{cases}$$

It is easy to see that $\Gamma_+(\tilde{f}_s) \subset \Gamma_+(f_s)$ for $s \neq 0$. Then by Corollary 7

$$\mu^{\text{gen}}(\tilde{f}_s) \geq \mu^{\text{gen}}(f_s) = d^2 - 4d + 2.$$

But by formula (K) we obtain

$$\mu^{\text{gen}}(\tilde{f}_s) = d^2 - 4d + 1,$$

a contradiction.

Ib. $(i, j) = (0, d-2)$ or $(i, j) = (d-2, 0)$. Both cases are similar, so we will consider only the case $(i, j) = (0, d-2)$. We define the auxiliary singularity

$$\tilde{f}_0(x, y) := y^{d-2} + x^d.$$

By formula (K) $\mu(\tilde{f}_0) = d^2 - 4d + 3$ and obviously $\Gamma_+(\tilde{f}_0) \subset \Gamma_+(f_s)$ for $s \neq 0$. Hence $\tilde{f}_s(x, y) := f_s(x, y) - y^d + \alpha y^{d-2}$ for some generic $0 \neq \alpha \in \mathbb{C}$, would be a non-degenerate deformation of \tilde{f}_0 such that $\Gamma_+(\tilde{f}_s) = \Gamma_+(f_s)$ for $s \neq 0$. Hence $\mu^{\text{gen}}(\tilde{f}_s) = d^2 - 4d + 2$. This gives $\mu(\tilde{f}_0) - \mu^{\text{gen}}(\tilde{f}_s) = 1$, that is the first jump of Milnor numbers for the singularity $\tilde{f}_0(x, y) = y^{d-2} + x^d$ is equal to 1. This is impossible by Bodin result ([Bod07], Section 7) because $d \geq 6$ and $\text{GCD}(d-2, d) = 2$ (he proved that the first jump for non-degenerate deformations is equal to 2 in this case).

II. $\text{ord}_{(x,y)} f_s > d-2$. Then $\Gamma_+(f_s) \subset \Gamma_+(x^{d-1} + y^{d-1})$. Hence $\mu^{\text{gen}}(f_s) \geq \mu(x^{d-1} + y^{d-1}) = (d-2)^2 = d^2 - 4d + 4$, which contradicts the supposition that $\mu^{\text{gen}}(f_s) = d^2 - 4d + 2$.

B. Assume now that f_0 is non-convenient i.e. $f_0(x, y) = x\tilde{f}_0(x, y)$ (case I) or $f_0(x, y) = y\tilde{f}_0(x, y)$ (case II) or $f_0(x, y) = xy\tilde{f}_0(x, y)$ (case III), where \tilde{f}_0 is convenient of degree $d-1$ and $\tilde{\tilde{f}}_0$ is convenient of degree $d-2$. Take any integer $k \leq (d-1)(d-2)$ and consider cases:

1. $k \neq d^2 - 4d + 2$ or d odd. Then there exists a deformation (f_s^1) of $x^d + y^d$ such that $\mu^{\text{gen}}(f_s^1) = k$. Let $f_s^2 := f_s^1 - x^d - y^d$. Then for the deformation $f_s := f_0 + f_s^2 + sx^d$ in case I or $f_s := f_0 + f_s^2 + sy^d$ in case II or $f_s := f_0 + f_s^2 + sx^d + sy^d$ in case III we obviously have $\mu^{\text{gen}}(f_s) = k$.

2. $k = d^2 - 4d + 2$, d is even and $d \geq 6$. For the deformation $f_s(x, y) := f_0(x, y) + sy^{d-1} + sx^2 y^{d-4} + sx^{d+2}$ in case I and III and $f_s(x, y) := f_0(x, y) + sx^{d-1} + sy^2 x^{d-4} + sy^{d+2}$ in case II (the summands sx^{d+2} and sy^{d+2} are superfluous; they have been added in order to use formula (K)) we have

$$\mu^{\text{gen}}(f_s) = d^2 - 4d + 2.$$

This ends the proof of Theorem 1.

3 Proof of Theorem 2

For any holomorphic function germs f, g at $0 \in \mathbb{C}^2$ by $i_0(f, g)$ we will denote the *intersection multiplicity* of the plane curve singularities $f = 0$ and $g = 0$ at 0. Since $\mathcal{M}^{\text{nd}}(f_0)$ is a subsequence of $\mathcal{M}(f_0)$, it suffices to prove that the number $(d-1)(d-2) + 1$ and the number $d^2 - 4d + 2$ for $d \geq 6$ are generic Milnor numbers of some deformations of f_0 . Consider first the number $(d-1)(d-2) + 1$. Let $f_0 = L_1 \cdots L_d$ be a factorization of f_0 into linear forms (no pair of them are proportional). We define the deformation of f_0 by

$$f_s = sL_1^{d-1} + f_0.$$

Take $s \neq 0$. Without loss of generality we may assume that $L_1(x, y) = x$. Then $f_0(x, y) = x(\alpha y^{d-1} + \dots)$ where $\alpha \neq 0$. Hence

$$\begin{aligned}\mu(f_s) &= \mu(sx^{d-1} + x(\alpha y^{d-1} + \dots)) = \\ &= i_0((d-1)sx^{d-2} + (\alpha y^{d-1} + \dots) + x \frac{\partial(\alpha y^{d-1} + \dots)}{\partial x}, x((d-1)\alpha y^{d-2} + \dots)) = \\ &= i_0((d-1)sx^{d-2} + (\alpha y^{d-1} + \dots) + x \frac{\partial(\alpha y^{d-1} + \dots)}{\partial x}, x) + \\ &\quad + i_0((d-1)sx^{d-2} + (\alpha y^{d-1} + \dots) + x \frac{\partial(\alpha y^{d-1} + \dots)}{\partial x}, (d-1)\alpha y^{d-2} + \dots) = \\ &= (d-1) + (d-2)(d-2) = (d-1)(d-2) + 1.\end{aligned}$$

Consider now the number $d^2 - 4d + 2$ for $d \geq 6$. Let $f_0 = L_1 \cdots L_d$ be a factorization of f_0 into linear forms. We may assume that $L_1(x, y) = \alpha x + \beta y$ where $\alpha \neq 0$. If we take a linear change of coordinates $\Phi : x' = L_1(x, y)$, $y' = y$ in \mathbb{C}^2 then the homogeneous singularity $\tilde{f}_0(x', y') := f_0 \circ \Phi^{-1}(x', y') = x' f_1(x', y')$ is non-convenient and degree d . Then by Theorem 1 there exists a deformation $(\tilde{f}_s)_{s \in S}$ of \tilde{f}_0 such that $\mu^{\text{gen}}(\tilde{f}_s) = d^2 - 4d + 2$. Hence for the deformation $f_s := \tilde{f}_s \circ \Phi$, $s \in S$, of f_0 we obtain $\mu(f_s) = \mu(\tilde{f}_s \circ \Phi) = \mu(\tilde{f}_s) = d^2 - 4d + 2$ for $s \neq 0$. Then $\mu^{\text{gen}}(f_s) = d^2 - 4d + 2$.

4 Proof of Theorem 3

Let $f_0(x, y) = x^d + y^d$, $d \geq 2$. Let us take a deformation (f_s) of f_0 which realizes the generic Milnor number μ_1 of f_0 i.e.

$$\mu^{\text{gen}}(f_s) < \mu(f_0) \quad (6)$$

and

$$\mu(f_0) - \mu^{\text{gen}}(f_s) \quad (7)$$

is minimal non-zero integer among all deformation of f_0 . In order to apply Płoski Theorem 8 to elements f_s , $s \neq 0$, of the family (f_s) we have to fulfill the assumptions of this theorem. We will achieve this by modifying the deformation (f_s) to another one (\tilde{f}_s) which satisfies all the requested conditions. The first step is to reduce holomorphic f_s to polynomials (in variables x, y). Notice (f_s) , $s \neq 0$, is a μ -constant family. So, if we omit in $f_s(x, y)$ all the terms of order $> \mu^{\text{gen}}(f_s) + 1$ then we obtain a deformation (\tilde{f}_s) of f_0 such that

$$\text{ord}(f_s - \tilde{f}_s) > \mu^{\text{gen}}(f_s) + 1 = \mu(f_s) + 1, \quad s \neq 0.$$

Hence by well-known theorem ([AGZV85], Prop. 1 and 2 in Section 5.5, [Pł85], Prop. 1.2 and Lemma 1.4)

$$\mu(f_s) = \mu(\tilde{f}_s) \quad \text{for } s \neq 0.$$

This implies $\mu^{\text{gen}}(f_s) = \mu^{\text{gen}}(\tilde{f}_s)$. By this step we may assume in the sequel that the deformation (f_s) of f_0 which realizes μ_1 consists of polynomials.

The second step is to reduce the degree of f_s to d . For this we apply the method of Gabrielov and Kouchnirenko [GK75]. Notice first that there are terms in f_s of order $< d$ with non-zero coefficients. In fact if $\text{ord}(f_s) \geq d$ then $\mu(f_s) \geq \mu(x^d + y^d) = (d-1)^2$, a contradiction. Let

$$f_s(x, y) = x^d + y^d + u_1(s)x^{\alpha_1}y^{\beta_1} + \dots + u_k(s)x^{\alpha_k}y^{\beta_k},$$

where $u_i(s)$, $i = 1, \dots, k$, are non-zero holomorphic functions in a neighbourhood of $0 \in \mathbb{C}$, $u_i(0) = 0$. Denote $d_i := \alpha_i + \beta_i$, $\gamma_i := \text{ord}(u_i) > 0$, $u_i(s) = a_i s^{\gamma_i} + \dots$, $a_i \neq 0$.

By the above there exist d_i for which $d_i < d$. Let $N := \text{LCM}(\gamma_i)$, $v := \max \left(\frac{N \cdot (d - d_i)}{\gamma_i} \right)$. Then $v > 0$. We define a new holomorphic deformation of f_0 depending on two parameters

$$\begin{aligned}f_{s,t}(x, y) &:= \frac{f_s(t^N x, t^N y)}{t^{Nd}} \\ &= x^d + y^d + (a_1 s^{\gamma_1} + t \tilde{u}_1(s, t)) t^{v\gamma_1 + N(d_1 - d)} x^{\alpha_1} y^{\beta_1} + \dots\end{aligned}$$

for some holomorphic functions $\tilde{u}_i(s, t)$, $i = 1, \dots, k$. By semi-continuity of Milnor numbers in families of singularities we obtain that for any fixed $s \in S$, $s \neq 0$,

$$\mu(f_{s,0}) \geq \mu(f_{s,t}) \quad \text{for sufficiently small } t.$$

But for any fixed $s, t \neq 0$ sufficiently small

$$\mu(f_{s,t}) = \mu\left(\frac{f_{t^N s}(t^N x, t^N y)}{t^{Nd}}\right) = \mu(f_{t^N s}(x, y)) = \mu_1.$$

Hence $\mu(f_{s,0}) \geq \mu_1$. But

$$f_{s,0}(x, y) = x^d + y^d + \sum_{\substack{j \\ v\gamma_j + N(d_j - d) = 0}} a_j s^{\gamma_j} x^{\alpha_j} y^{\beta_j}.$$

Of course $d_j < d$ for j satisfying $v\gamma_j + N(d_j - d) = 0$. So, we have obtained a new deformation $\tilde{f}_s := f_{s,0}$ of f_0 for which $\deg \tilde{f}_s = d$, $\text{ord}(\tilde{f}_s) < d$ and $\mu^{\text{gen}}(\tilde{f}_s) \geq \mu_1$. By definition of μ_1 we have either $\mu^{\text{gen}}(\tilde{f}_s) = \mu_1$ or $\mu^{\text{gen}}(\tilde{f}_s) = \mu(f_0)$. The latter case is impossible because then (\tilde{f}_s) for $s \in S$ would be a μ -constant family in which one element is equal to f_0 . Since it is a family of plane curve singularities, the orders of this singularities are the same. Hence

$$\text{ord}(f_0) = \text{ord}(\tilde{f}_s) \quad \text{for } s \in S,$$

which is impossible.

Summing up, we have obtained a deformation (\tilde{f}_s) of f_0 for which $\mu^{\text{gen}}(\tilde{f}_s) = \mu_1$, $\deg(\tilde{f}_s) = d$, $\text{ord}(\tilde{f}_s) < d$, $s \neq 0$, and moreover d -th homogeneous component of \tilde{f}_s is equal to f_0 . So, for any fixed $s \neq 0$ \tilde{f}_s satisfies the assumption of Theorem 8. By this theorem

$$\mu_1 = \mu(\tilde{f}_s) \leq (d-1)^2 - \left\lfloor \frac{d}{2} \right\rfloor.$$

Now we prove the opposite inequality

$$\mu_1 \geq (d-1)^2 - \left\lfloor \frac{d}{2} \right\rfloor. \quad (8)$$

It suffices to give deformations (f_s) of f_0 for which $\mu^{\text{gen}}(f_s) = (d-1)^2 - \lfloor d/2 \rfloor$. Consider two cases:

1. d is even i.e. $d = 2k$, $k \geq 1$. We define a deformation of f_0 by

$$f_s(x, y) := x^d + (y^2 + sx)^{\frac{d}{2}} = x^{2k} + (y^2 + sx)^k.$$

Then we easily find for $s \neq 0$

$$\mu(f_s) = i_0(2kx^{2k-1} + ks(y^2 + sx)^{k-1}, 2ky(y^2 + sx)^{k-1}) = 4k^2 - 5k + 1 = (d-1)^2 - \left\lfloor \frac{d}{2} \right\rfloor.$$

2. d is odd i.e. $d = 2k + 1$, $k \geq 1$. We define a deformation of f_0 by

$$f_s(x, y) := (x + y) \prod_{i=1}^k (sw_i(x + y) + x^2 + 2\Re(\varepsilon^i)xy + y^2), \quad (9)$$

where $\varepsilon = \exp(\frac{2\pi i}{2k+1})$ is a primitive root of unity of degree $2k + 1$ and $w_i = \frac{1 - \Re(\varepsilon^i)}{1 - \Re(\varepsilon)}$, $i = 1, \dots, k$. We easily check that (f_s) is a deformation of $f_0(x, y) = x^{2k+1} + y^{2k+1}$. Moreover if we denote by $\tilde{f}_0, \tilde{f}_1, \dots, \tilde{f}_k$ the successive factors in (9) we easily compute that $i_0(\tilde{f}_0, \tilde{f}_i) = 2$, $i = 1, \dots, k$, and $i_0(\tilde{f}_i, \tilde{f}_j) = 4$, $i, j = 1, \dots, k$, $i \neq j$. Hence by a well-known formula for the Milnor number of a product of singularities we obtain

$$\mu(\tilde{f}_0 \cdots \tilde{f}_k) = \sum_{i=0}^k \mu(\tilde{f}_i) + 2 \sum_{\substack{i,j=0 \\ i < j}}^k i_0(\tilde{f}_i, \tilde{f}_j) - k = 4k^2 - k = (d-1)^2 - \left\lfloor \frac{d}{2} \right\rfloor.$$

5 Proof of Theorem 4

Let us begin with a remark. The second part of Theorem 4 concerns only homogeneous singularities of degree $d \geq 5$. For degrees $d = 2, 3, 4$ the sequences $\mathcal{M}(f_0)$ do not depend on the coefficients of f_0 and they are as follows

$$\mathcal{M}(f_0) = \begin{cases} (1, 0) & \text{for } d = 2, \\ (4, 3, 2, 1, 0) & \text{for } d = 3, \\ (9, 7, 6, 5, 4, 3, 2, 1, 0) & \text{for } d = 4. \end{cases}$$

For $d = 2, 3$ it is an easy fact and for $d = 4$ it follows from [BK14]. Moreover, by Theorem 4 we obtain that for $d = 5$ we have only two possibilities

$$\mathcal{M}(f_0) = \begin{cases} (16, 13, 12, \dots, 1, 0) & \text{for } f_0 \text{ with generic coefficients,} \\ (16, 14, 13, 12, \dots, 1, 0) & \text{otherwise.} \end{cases}$$

Now we may pass to the proof of Theorem 4. For the first part of the theorem we repeat the reasoning in the proof of inequality $\mu_1(x^d + y^d) \leq (d-1)^2 - [d/2]$ in Theorem 3 because it works for any homogeneous singularity.

For the second part of the theorem let $f_0(x, y) = c_0 y^d + c_1 y^{d-1} x + \dots + c_d x^d$ be an arbitrary homogeneous singularity of degree $d \geq 5$. We may assume that $c_0 = 1$ (because singularities for which $c_0 = 0$ are „not generic“). Denote

$$f_0^c(x, y) := y^d + c_1 y^{d-1} x + \dots + c_d x^d, \quad c = (c_1, \dots, c_d) \in \mathbb{C}^d.$$

By the first part of the theorem $\mu_1(f_0^c) \leq (d-1)^2 - [d/2]$. We will find a polynomial $F(c_1, \dots, c_d)$, $F \neq 0$, such that if

$$\mu_1(f_0^c) = (d-1)^2 - \left\lceil \frac{d}{2} \right\rceil \quad (10)$$

then $F(c) = 0$. This will give the second part of the theorem and finish the proof.

Let us take an arbitrary f_0^c for which (10) holds. Let

$$f_0^c(x, y) = (y - a_1 x) \dots (y - a_d x), \quad a_i \neq a_j \text{ for } i \neq j$$

be the factorization of f_0^c into linear parts. We will also denote this polynomial by f_0^a , where $a = (a_1, \dots, a_d)$ and call a_1, \dots, a_d the roots of f_0^a . By the Vieta formulas connecting c_i with a_j it suffices to find a polynomial $G(a_1, \dots, a_d)$, $G \neq 0$, such that if $\mu_1(f_0^a) = (d-1)^2 - [d/2]$, then $G(a) = 0$. So, take f_0^a for which $\mu_1(f_0^a) = (d-1)^2 - [d/2]$. Using the same method as in the proof of Theorem 3 there exists a deformation (f_s) of f_0^a such that for $s \neq 0$ sufficiently small

1. $\deg f_s = d$,
2. d -th homogeneous component of f_s is equal to f_0^a ,
3. $\mu^{\text{gen}}(f_s) = (d-1)^2 - [d/2]$.

Let us fix $s \neq 0$. By Theorem 9 we obtain a factorization

$$f_s = L Q_1 \dots Q_{[d/2]},$$

where L is either a linear form (if d is odd) or $L = 1$ (if d is even), Q_i are irreducible polynomials of degree 2, L has a common tangent with each Q_i (if d is odd) and $i_0(Q_i, Q_j) = 4$ for $i \neq j$. Since Q_i is irreducible, $Q_i = L_i + \tilde{Q}_i$ where L_i is a non-zero linear form and \tilde{Q}_i is a non-zero quadratic form. Moreover, the equality $i(Q_i, Q_j) = 4$ for $i \neq j$ implies the all L_i are proportional. Additionally in the odd case L is also proportional to L_i . Notice also that d -th homogeneous component of $L Q_1 \dots Q_{[d/2]}$ is equal to $L \tilde{Q}_1 \dots \tilde{Q}_{[d/2]}$, which by above condition 2 implies $f_0^a = L \tilde{Q}_1 \dots \tilde{Q}_{[d/2]}$. Now we consider the cases:

I. d is odd. By renumbering a_1, \dots, a_d we may assume that $L = (y - a_1 x)$, $\tilde{Q}_1 = (y - a_2 x)(y - a_3 x), \dots, \tilde{Q}_{[d/2]} = (y - a_{d-1} x)(y - a_d x)$. Let $L_i = w_i L$, $w_i \in \mathbb{C} \setminus \{0\}$, $i = 1, \dots, [d/2]$. Then from the condition $i_0(Q_i, Q_j) = 4$, $i \neq j$, we obtain in particular for $i = 1$, $j = 2$

$$4 = i_0(w_1 L + \tilde{Q}_1, w_2 L + \tilde{Q}_2) = i_0(\tilde{Q}_1 - \frac{w_1}{w_2} \tilde{Q}_2, w_2 L + \tilde{Q}_2).$$

Since $\tilde{Q}_1 - \frac{w_1}{w_2}\tilde{Q}_2$ is a form of degree 2, we get $\tilde{Q}_1 - \frac{w_1}{w_2}\tilde{Q}_2 = uL^2$ for some $u \in \mathbb{C} \setminus \{0\}$. Hence there exist $z_1, z_2 \in \mathbb{C} \setminus \{0\}$ such that

$$z_1\tilde{Q}_1 + z_2\tilde{Q}_2 = L^2.$$

Then the non-zero z_1, z_2 satisfy the system of equations

$$\begin{aligned} z_1 + z_2 &= 1, \\ (a_2 + a_3)z_1 + (a_4 + a_5)z_2 &= 2a_1, \\ a_2a_3z_1 + a_4a_5z_2 &= a_1^2. \end{aligned}$$

Hence

$$\begin{vmatrix} 1 & 1 & 1 \\ a_2 + a_3 & a_4 + a_5 & 2a_1 \\ a_2a_3 & a_4a_5 & a_1^2 \end{vmatrix} = 0.$$

We have obtained a non-trivial relation $\tilde{G}(a_1, \dots, a_5) = 0$ between the roots a_1, \dots, a_5 . So if we put

$$G(a_1, \dots, a_d) := \prod_{\sigma \in V_d^5} \tilde{G}(a_{\sigma(1)}, \dots, a_{\sigma(5)}),$$

where V_d^k denotes the set of all partial permutations of length k from a d -set, then G is a non-zero polynomial in $\mathbb{C}[a_1, \dots, a_d]$ such that if $G(a) \neq 0$ then $\mu_1(f_0^a) < (d-1)^2 - [d/2]$. This ends the proof of the theorem in this case.

II. d is even. Then $d \geq 6$. By renumbering a_0, \dots, a_d we may assume that $\tilde{Q}_1 = (y - a_1x)(y - a_2x), \dots, \tilde{Q}_{[d/2]} = (y - a_{d-1}x)(y - a_dx)$. Let $L_i = w_iL$, $w_i \in \mathbb{C} \setminus \{0\}$, $i = 1, \dots, [d/2]$, where L is a fixed non-zero linear form. Repeating the reasoning as in I for the equality $i_0(Q_1, Q_2) = 4$ we get that there exist $z_1, z_2 \neq 0$ such that

$$z_1\tilde{Q}_1 + z_2\tilde{Q}_2 = L^2.$$

We may assume that either $L = x$ or $L = y$ or $L = y - \alpha x$, $\alpha \neq 0$. In the first two cases we easily obtain, as in I, non-trivial relations $G_1(a_1, a_2, a_3, a_4) = 0$ and $G_2(a_1, a_2, a_3, a_4) = 0$, respectively, between the roots a_1, \dots, a_4 . In the third case we obtain the relation

$$((a_3 + a_4) - (a_1 + a_2))\alpha^2 - 2(a_3a_4 - a_1a_2)\alpha + (a_1 + a_2)a_3a_4 - a_1a_2(a_3 + a_4) = 0$$

between a_1, a_2, a_3, a_4 and α . But if we apply the same reasoning to the equality $i_0(Q_2, Q_3) = 4$ we obtain a second relation

$$((a_5 + a_6) - (a_3 + a_4))\alpha^2 - 2(a_5a_6 - a_3a_4)\alpha + (a_3 + a_4)a_5a_6 - a_3a_4(a_5 + a_6) = 0$$

between a_3, a_4, a_5, a_6 and the same α . Hence the resultant of these two polynomials with respect to α must be equal to 0, which gives a non-trivial relation $G_3(a_1, \dots, a_6) = 0$ between roots a_1, \dots, a_6 . Hence

$$G(a_1, \dots, a_d) := \prod_{\sigma \in V_d^4} (G_1G_2)(a_{\sigma(1)}, \dots, a_{\sigma(4)}) \prod_{\sigma \in V_d^6} G_3(a_{\sigma(1)}, \dots, a_{\sigma(6)})$$

is a non-trivial polynomial such that if $G(a) \neq 0$ then $\mu_1(f_0^a) < (d-1)^2 - [d/2]$. This ends the proof of the theorem in this case.

6 Concluding remarks

We have completely solved the problem of possible generic Milnor numbers of all non-degenerate deformations of homogeneous plane singularities. The same problem for the family of all deformations is more complicated. In the particular case $f_0(x, y) = x^d + y^d$, $d \geq 2$, we have only found $\mu_1 = (d-1)^2 - [d/2]$. For generic homogeneous singularities of degree d this is not longer true by Theorem 4. We do not know the exact value of μ_1 in this generic case. We only conjecture that for generic homogeneous singularities of degree d $\mu_1 = (d-1)^2 - (d-2)$. If it is true then by Theorem 2 we would get the whole sequence $\mathcal{M}(f_0)$ in this case.

Conjecture *If f_0 is a homogeneous singularity of degree d with generic coefficients then*

$$\mathcal{M}(f_0) = ((d-1)^2, (d-1)^2 - (d-2), \dots, 1, 0).$$

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